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Kinematics of homogeneous axisymmetric turbulence

By ERIK LINDBORG

Department of Mechanics, KTH, 100 44 Stockholm, Sweden

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It is shown that the expressions for the correlation tensors of homogeneous axisymmetric turbulence can be considerably simplified compared to previous analyses of Batchelor (1946) and Chandrasekhar (1950). Representations of the axisymmetric two-point correlations tensors are found, such that each measurable correlation corresponds to a single scalar function, and moreover such that the equations of continuity relating different tensor components to each other take the most simple form. Reflectional symmetry in planes normal to but not in planes through the axis of symmetry is demanded, which allows a full description of states with rotation about the axis of symmetry. The second and third-order velocity correlation tensors and the first-order pressure–velocity correlation tensor are analysed with the new method. Small separation expansions of the correlation functions yield the quantities which have to be measured to determine various terms in the governing equations for the Reynolds stresses and the dissipation tensor. A scalar Poisson equation for the pressure–strain is derived, and the single-point solution is written as a sum of integrals over measurable correlation functions. The simplified analysis can be of great experimental importance. It reveals in a simple way how a full experimental picture of homogeneous axisymmetric turbulence can be obtained by measuring components of the velocity at two points at variable distance from each other on a line perpendicular to the mean flow in a wind tunnel. By using the Fourier–Bessel transform it is also shown that the three-dimensional energy, transfer, and pressure–strain spectra can be extracted from such measurements.

1. Introduction

The study of axisymmetric turbulence is particularly interesting since it is the simplest form of turbulence in which effects of pressure-strain, anisotropy distribution among different scales and return to isotropy can be studied. It is relatively easy to realize in a wind tunnel experiment and is attractive from a theoretical point of view because of its simplicity.

The investigation of how different correlations relate to each other under the symmetry constraint is fundamental to the theory of axisymmetric turbulence. Batchelor (1946) analysed the second-order two-point correlation tensor

$$R_{ij}(\boldsymbol{x},\boldsymbol{r}) = \langle u_i(\boldsymbol{x})u_j(\boldsymbol{x}+\boldsymbol{r})\rangle \tag{1}$$

for the case of homogeneous axisymmetric turbulence using the method of invariants expounded by Robertson (1940). Homogeneity means that $\boldsymbol{R}$ is independent of position $\boldsymbol{x}$ and also implies the index symmetry (Batchelor 1946)

$$R_{ij}(\boldsymbol{r}) = R_{ji}(-\boldsymbol{r}). \tag{2}$$

The continuity equation for incompressible fluids implies that $\boldsymbol{R}$ is solenoidal with respect to both its indices,

$$\frac{\partial R_{ij}}{\partial r_i} = \frac{\partial R_{ij}}{\partial r_j} = 0. \tag{3}$$

Using these two properties of $\boldsymbol{R}$, Batchelor showed that in the case of axisymmetry $\boldsymbol{R}$ can be expressed as

$$R_{ij} = r_i r_j A + \delta_{ij} B + \lambda_i \lambda_j C + (\lambda_i r_j + \lambda_j r_i) D, \tag{4}$$

where $\boldsymbol{\lambda}$ is the unit vector in the symmetry direction and A, B, C and D are scalar functions of $r = |\,\boldsymbol{r}\,|$ and $\mu = \boldsymbol{r}\cdot\boldsymbol{\lambda}/r$, coupled by the two equations (Batchelor 1946, equations (2.14) and (2.15))

$$4A + r\frac{\partial A}{\partial r} + \frac{1}{r}\frac{\partial B}{\partial r} - \frac{\mu}{r^2}\frac{\partial B}{\partial \mu} + \mu\frac{\partial D}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial D}{\partial \mu} = 0, \tag{5a}$$

$$\frac{1}{r}\frac{\partial B}{\partial \mu} + \mu\frac{\partial C}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial C}{\partial \mu} + 4D + r\frac{\partial D}{\partial r} = 0. \tag{5b}$$

The representation (4) of the axisymmetric form of $\boldsymbol{R}$ is found by a straightforward application of the method of Robertson (1940).

Chandrasekhar (1950) developed a method based on the idea of writing an axisymmetric solenoidal tensor as the curl of a general axisymmetric skew tensor, i.e. a reflectionally non-invariant tensor. By using this method he could explicitly write $\boldsymbol{R}$ in terms of only two scalar functions Q_1 and Q_2. The relations between these two functions and Batchelor's four functions are (Chandrasekhar 1950, equations (50))

$$A = (\mathrm{D}_r - \mathrm{D}_{\mu\mu})Q_1 + \mathrm{D}_r Q_2, \tag{6a}$$

$$B = [-(r^2\mathrm{D}_r + r\mu\mathrm{D}_\mu + 2) + r^2(1-\mu^2)\mathrm{D}_{\mu\mu} - r\mu\mathrm{D}_\mu]Q_1 - [r^2(1-\mu^2)\mathrm{D}_r + 1]Q_2, \tag{6b}$$

$$C = -r^2\mathrm{D}_{\mu\mu}Q_1 + (r^2\mathrm{D}_r + 1)Q_2, \tag{6c}$$

$$D = (r\mu\mathrm{D}_\mu + 1)\mathrm{D}_\mu Q_1 - r\mu\mathrm{D}_r Q_2, \tag{6d}$$

where

$$\mathrm{D}_r = \frac{1}{r}\frac{\partial}{\partial r} - \frac{\mu}{r^2}\frac{\partial}{\partial \mu}, \quad \mathrm{D}_\mu = \frac{1}{r}\frac{\partial}{\partial \mu}, \quad \mathrm{D}_{\mu\mu} = \mathrm{D}_\mu \mathrm{D}_\mu. \tag{7}$$

Chandrasekhar also applied this type of formalism to the third-order two-point triple correlation tensor and wrote it in terms of six scalar functions, with the differential operators in (7) operating on them.

It is rather intricate to get a deeper insight from Batchelor's representation into how various correlations between different velocity components exactly relate to each other. None of the four functions of (4) alone represents a correlation which can be individually measured. The form of equations (5*a,b*) admits no explicit solution for any pair of the four functions in terms of the other pair. Despite the mathematical elegance of Chandrasekhar's representation the problem becomes even worse if we consider (6*a–d*) instead of Batchelor's equations.

The question is now if any simpler representation of axisymmetric turbulence can be found, and the answer is 'yes'. The axisymmetric correlation tensors can be represented by scalar functions corresponding to correlations which can be individually measured, and the equations of continuity can at the same time be considerably simplified. This will be clear in the case of the first-order pressure–velocity correlation tensor, the second-order velocity correlation tensor and especially in the case of the third-order velocity correlation tensor, which would be difficult to analyse using either the method of Batchelor or the method of Chandrasekhar.

Batchelor (1946) and Chandrasekhar (1950) analysed axisymmetry including total reflectional symmetry. This excludes states with rotation. Here we will take the more general approach of only demanding reflectional symmetry in planes normal to the symmetry axis, which allows rotation about the same axis.

1.1. *Experimental background*

This work is mainly motivated by experimental needs. To determine the two components of the axisymmetric dissipation tensor, measurements were performed in the new wind tunnel of KTH (Royal Institute of Technology). For this purpose two hot-wire cross-probes simultaneously measured two velocity components at two points at variable distance on a line perpendicular to the mean flow. By using Taylors's hypothesis in the mean flow direction it was possible to construct two-dimensional correlation surfaces whose curvatures could be estimated at the origin. An automatic traversing system made it possible to organize relatively long series of measurements without any human interference. The question emerged of how much information can be extracted from such measurements. Is it possible to determine not only the components of the dissipation tensor but also the different terms in the dynamical equation for this tensor, if similar measurements of the triple correlations can be carried out? Is it possible to obtain full three-dimensional energy and transfer spectra from such measurements, to be compared with results from direct numerical simulations (DNS) or from dynamical two-point modelling? Is it possible to solve the Poisson equation for the pressure–strain in terms of correlation functions which can be obtained by such measurements? The answers to these questions are in general affirmative, as we shall see, and the present analysis gives the relations between the measurable correlation functions and the various quantities of fundamental interest. These relations, which would be very complicated to derive using either the method of Batchelor or the method of Chandrasekhar, have already proved to be valuable for the analysis of the ongoing experiments in the wind tunnel at KTH (A. Johansson & T. Sjögren, Private communication). Such experiments are not only crucial when answers are sought to fundamental questions of turbulence, such as Kolmogorov's hypothesis of isotropy of small scales, but can also be of great importance for validation of single-point turbulence models designed for technical use, such as the model of the rapid pressure–strain by Johansson & Hallbäck (1994). The new analysis opens a door to measurements at high Reynolds numbers of quantities such as the rapid pressure–strain, which formerly have only been evaluated at low Reynolds numbers by use of DNS (see for example Lee & Reynolds 1985).

In this context it is also interesting to note that a lot of theoretical and computational work has recently been performed on homogeneous turbulence subjected to mean rotation. Cambon & Jacquin (1989) used a phenomenological two-point closure to make a detailed investigation of three-dimensional spectral energy transfer in this type of turbulence. Bartello, Métais & Lesieur (1994) investigated coherent structures and two-dimensionality of rotating turbulence by DNS. Johansson, Hallbäck & Lindborg (1994) analysed the difficulties associated with mean rotation in single-point modelling. These works are only some examples of the efforts which have recently been made in this field, efforts which make it an urgent task also to perform experiments on this type of turbulence. That such experiments can indeed

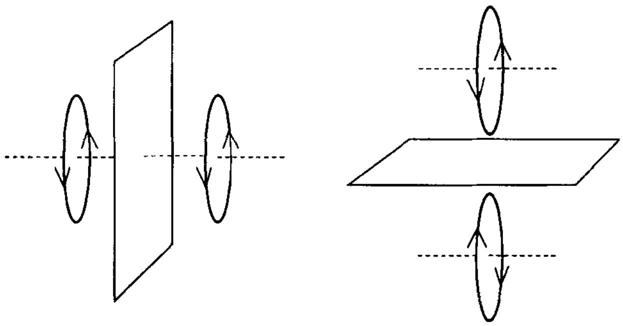

FIGURE 1. A rotation is invariant under reflection in a plane normal to the axis of rotation, but not in a plane parallel to the axis of rotation.

be carried out has recently been shown by Jacquin *et al.* (1990) and Leuchter & Dupeuble (1993). In the present analysis the skew tensors which are needed for a description of a flow with mean rotation are included. The inclusion of these tensors enables the experimentalist to use the present analysis in exactly the same way in the case of rotating turbulence as in the non-rotating case. With the same type of experiments the same things can be achieved in the two cases.

Homogeneous axisymmetric turbulence is the first test case for hypotheses on general effects of anisotropy. Experiments on this form of turbulence are of great interest. A simple and concise outline of the kinematics of homogeneous axisymmetric turbulence is therefore needed. It is the ambition of the present work to fill this need.

2. Axisymmetry

A homogeneous axisymmetric physical state is invariant under rotations around axes that are normal to a plane, and invariant under translations along the same axes. In the following we will need a unit vector λ which is normal to the given plane. There are two possible directions of λ. In an experimental situation we can fix λ with reference to some external direction in the experimental environment. In a wind tunnel experiment on homogeneous axisymmetric turbulence we can define λ as the unit vector in the mean flow direction, but the opposite direction would do just as well. In the case of turbulence in a rotating container, as for example the apparatus in the experiment of Ibbetson & Tritton (1975), we can define λ as either of two possible unit vectors aligned with the axis of rotation.

In the definition of axisymmetry we include reflectional symmetry in planes normal to λ, but not reflectional symmetry in planes containing λ. The last type of reflectional symmetry is broken if there is rotation about λ, while this is not true for the first type (see figure 1). When both these symmetries occur we will say that we have 'strong axisymmetry' in distinction to the weaker type of symmetry which is the case if there is rotation about λ.

Turbulence obeying the weaker, but not the stronger kind of axisymmetry can be generated by letting a rotating fluid pass through a grid. The fluid can be set in rotation by keeping it in a rotating apparatus, which could be a section of a wind tunnel, as in the experiment of Jacquin *et al.* (1990), or a container, as in the experiment of Ibbetson & Tritton (1975). A homogeneous field is probably most easily produced in a wind tunnel. Depending on whether the measuring instrument takes part in the rotational motion or not, it can most appropriately be described either as a system rotation or a mean flow rotation. In the first case the rotation enters into the turbulence dynamical equation as a Coriolis term, and in the second case as a mean flow gradient term. In each mode of description the rotational term breaks the reflectional symmetry in planes parallel to $\boldsymbol{\lambda}$ (axis of rotation), but not in planes normal to $\boldsymbol{\lambda}$. Hence, if there is nothing else in the experimental configuration that breaks any axisymmetry condition, including reflectional symmetry in planes normal to $\boldsymbol{\lambda}$, then the generated turbulence will obey the weaker, but not the stronger kind of axisymmetry. Here we will restrict ourselves to describe rotation as a rotation of the mean flow.

3. Second-order correlation tensor

According to the method of invariants† the nth-order two-point correlation tensor of homogeneous axisymmetric turbulence can be represented by a sum of all linearly independent nth-order tensors that can be formed from the separation vector $\boldsymbol{r}$ between the two points which we denote by O and P, the unit vector $\boldsymbol{\lambda}$, the Kronecker tensor and the permutation tensor ϵ_{ijk}. Each of the tensors multiplies a scalar which is only a function of the length of $\boldsymbol{r}$ and the angle between $\boldsymbol{r}$ and $\boldsymbol{\lambda}$. In order to find the most convenient representation of the second-order correlation tensor we form the orthogonal unit vectors

$$\boldsymbol{e}^{(1)} = \frac{1}{\rho}\boldsymbol{\lambda} \times \boldsymbol{r}, \quad \boldsymbol{e}^{(2)} = \boldsymbol{e}^{(1)} \times \boldsymbol{\lambda}, \tag{8}$$

where $\rho = \mid \boldsymbol{r} \times \boldsymbol{\lambda} \mid$. These relations can also be expressed in Cartesian tensor notation, using the permutation tensor. The vector $\boldsymbol{r}$ is not defined as a position vector of a single point with reference to an origin in a coordinate system, but as the separation vector between two measurement points in the turbulence field, and thus it is coordinate-system independent, i.e. it is a true vector. Since $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ are obtained by vector multiplications of $\boldsymbol{\lambda}$ and $\boldsymbol{r}$, they are also vectors.

From the three unit vectors $\boldsymbol{\lambda}$, $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ it is now possible to form nine independent second-order tensors, each of which can be multiplied by a scalar function. By virtue of the index symmetry condition (2) only six of the scalar functions will be independent. If we apply this condition the second-order correlation can be represented as

$$\begin{aligned} R_{ij}(\boldsymbol{r}) = {} & \lambda_i\lambda_j R_1(\rho,z) + e_i^{(2)}e_j^{(2)}R_2(\rho,z) + e_i^{(1)}e_j^{(1)}R_3(\rho,z) \\ & +\lambda_i e_j^{(2)}R_4(\rho,z) - e_i^{(2)}\lambda_j R_4(\rho,-z) + \lambda_i e_j^{(1)}S_1(\rho,z) - e_i^{(1)}\lambda_j S_1(\rho,-z) \\ & +e_i^{(2)}e_j^{(1)}S_2(\rho,z) + e_i^{(1)}e_j^{(2)}S_2(\rho,-z), \end{aligned} \tag{9}$$

where $R_1, R_2, \ldots S_2$ are scalar functions and $z = \boldsymbol{r}\cdot\boldsymbol{\lambda}$. Rather than r and μ we have chosen ρ and z as the arguments of our scalar functions. R_1, R_2 and R_3 are even in z owing to the symmetry condition (2). In Appendix A the two-point velocity correlation is represented in dyadic notation. Here we have adopted Cartesian tensor notation in order to keep close to most classical works on turbulence. Each of the scalar functions in (9) represents a correlation between two measurable velocity components. Let u, v and w be the axial, radial and azimuthal velocity components, that is the components in the directions of $\boldsymbol{\lambda}, \boldsymbol{e}^{(2)}$ and $\boldsymbol{e}^{(1)}$ respectively, as shown in

† The basic idea behind this method is given in Appendix A, where dyadic notation is adopted, instead of Cartesian tensor notation. A good presentation is also given by Batchelor (1953).

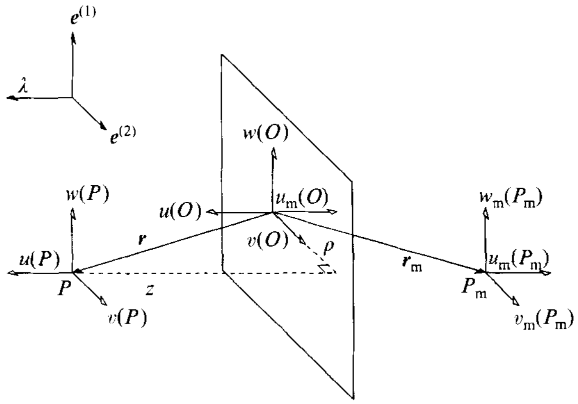

FIGURE 2. Reflection in a plane normal to the direction of symmetry.

figure 2. Then

$$\left.\begin{aligned} &R_1 = \langle u(O)u(P)\rangle, \quad R_2 = \langle v(O)v(P)\rangle, \quad R_3 = \langle w(O)w(P)\rangle, \quad R_4 = \langle u(O)v(P)\rangle, \\ &S_1 = \langle u(O)w(P)\rangle, \quad S_2 = \langle v(O)w(P)\rangle. \end{aligned}\right\} \tag{10}$$

Reflectional symmetry with respect to a plane normal to $\boldsymbol{\lambda}$ means that we can reflect the separation vector $\boldsymbol{r}$ and all the velocity components in this plane without changing $\boldsymbol{R}$. $\boldsymbol{\lambda}$ is chosen, with reference to some external direction, as one of two possible normal unit vectors to a plane. We can think of it as a vector being rigidly attached to an experimental apparatus and thus it must be kept fixed during the reflection. Let the subscript 'm' indicate the mirror image in the normal plane of $\boldsymbol{\lambda}$ through the point O, as in figure 2. Clearly we have $u_m = -u, v_m = v$ and $w_m = w$. By reflecting we find

$$R_4(\rho, z) = \langle u(O)v(P)\rangle = \langle u_m(O)v_m(P_m)\rangle = -R_4(\rho, -z), \tag{11}$$

$$S_1(\rho, z) = \langle u(O)w(P)\rangle = \langle u_m(O)w_m(P_m)\rangle = -S_1(\rho, -z), \tag{12}$$

$$S_2(\rho, z) = \langle v(O)w(P)\rangle = \langle v_m(O)w_m(P_m)\rangle = S_2(\rho, -z). \tag{13}$$

The corresponding relations for R_1, R_2 and R_3 are consistent with the fact that they are even functions of z. The same relations are obtained by reflection in any plane normal to $\boldsymbol{\lambda}$. The correlation tensor can now be written

$$\begin{aligned} R_{ij}(\boldsymbol{r}) = \lambda_i\lambda_j R_1 + e_i^{(2)}e_j^{(2)}R_2 + e_i^{(1)}e_j^{(1)}R_3 + (\lambda_i e_j^{(2)} + \lambda_j e_i^{(2)})R_4 \\ +(\lambda_i e_j^{(1)} + \lambda_j e_i^{(1)})S_1 + (e_i^{(2)}e_j^{(1)} + e_j^{(2)}e_i^{(1)})S_2. \end{aligned} \tag{14}$$

Reflectional symmetry in a plane containing the axis of symmetry, for example the $(\boldsymbol{\lambda}, \boldsymbol{r})$-plane implies that S_1 and S_2 are zero, which must be the case if there is no rotation about the axis of symmetry. All reasoning about reflectional symmetry can of course be carried out without any reference to velocity components. To see that a reflection in the $(\boldsymbol{\lambda}, \boldsymbol{e}^{(1)})$-plane has the same effect on $\boldsymbol{R}$ as a reflection in the $(\boldsymbol{\lambda}, \boldsymbol{r})$-plane we note that $\boldsymbol{e}^{(1)}(\boldsymbol{r}_m) = -\boldsymbol{e}_m^{(1)}(\boldsymbol{r}_m)$ and $\boldsymbol{e}^{(2)}(\boldsymbol{r}_m) = \boldsymbol{e}_m^{(2)}(\boldsymbol{r}_m)$ where the subscript 'm' now indicates the mirror image in either of these two planes.

The non-skew part of (14) can be rewritten in the same form as Batchelor's representation (4). To do so, substitute into (14) the identities

$$e_i^{(1)}e_j^{(1)} = \delta_{ij} - \lambda_i\lambda_j - e_i^{(2)}e_j^{(2)}, \tag{15a}$$

$$e_i^{(2)} = \frac{1}{r(1-\mu^2)^{1/2}}(r_i - r\mu\lambda_i). \tag{15b}$$

The relations between the first four functions of (14) and A, B, C and D can also be found by projection of (4) onto suitable combinations of two of our three unit vectors. This gives

$$\left.\begin{aligned} R_1 = r^2\mu^2 A + B + C + 2r\mu D, \quad R_2 = r^2(1-\mu^2)A + B, \\ R_3 = B, \quad R_4 = r^2\mu(1-\mu^2)^{1/2}A + r(1-\mu^2)^{1/2}D. \end{aligned}\right\} \tag{16}$$

The skew part of (14) can be rewritten in terms of tensors including a single permutation symbol, the Kronecker delta and the two given vectors $\boldsymbol{r}$ and $\boldsymbol{\lambda}$.

From simple symmetry arguments, R_1, R_2, R_3 and S_2 must be even in ρ while R_4 and S_1 must be odd in ρ. By a function being even or odd in ρ we mean a function whose Taylor expansion contains only even or odd powers of ρ. The evenness of for example R_1 is analogous to the evenness of the function $f(r)$ of the isotropic correlation tensor (see §3.1). Let $R_1'(x,z) = \langle u(0,0)u(x,z)\rangle$ be the measured correlation in a Cartesian laboratory frame system with origin at O, with z in the direction of the axis of symmetry and u the velocity component in the same direction. Then, assuming axisymmetry, it is obvious that R' must be even in x. Identifying $\boldsymbol{R}$ with the measurable correlation tensor it is also clear that for non-negative x we have $R_1'(x,z) = R_1(x,z)$. The evenness of R_1 in ρ now follows by changing x to ρ. The corresponding properties of the other correlation functions can be derived in the same way.

When applying the continuity condition (3) we use the following relations:

$$\frac{\partial}{\partial r_j}e_i^{(1)} = -\frac{1}{\rho}e_j^{(1)}e_i^{(2)}, \quad \frac{\partial}{\partial r_j}e_i^{(2)} = \frac{1}{\rho}e_j^{(1)}e_i^{(1)}, \tag{17}$$

$$\lambda_j\frac{\partial}{\partial r_j} = \frac{\partial}{\partial z}, \quad e_j^{(2)}\frac{\partial}{\partial r_j} = \frac{\partial}{\partial \rho}, \quad e_j^{(1)}\frac{\partial}{\partial r_j} = 0. \tag{18}$$

The last relation is of course only true if the operator acts on functions of only ρ and z. The continuity condition (3) applied to (14) yields

$$\frac{\partial}{\partial \rho}(\rho R_4) = -\rho\frac{\partial R_1}{\partial z}, \tag{19a}$$

$$R_3 = \frac{\partial}{\partial \rho}(\rho R_2) + \rho\frac{\partial R_4}{\partial z}, \tag{19b}$$

$$\frac{\partial}{\partial \rho}(\rho^2 S_2) = -\rho^2\frac{\partial S_1}{\partial z}. \tag{19c}$$

Equations (19*a*,*b*) are equivalent to Batchelor's equations (5*a*,*b*).

3.1. *Isotropic case*

Isotropy can be looked upon as a special case of axisymmetry, in which $\boldsymbol{\lambda}$ becomes a unit vector in an arbitrary direction. For isotropic turbulence $\boldsymbol{R}$ can be written (Kármán & Howarth 1938, see also Hinze 1975)

$$R_{ij}(\boldsymbol{r}) = u_{rms}^2\left[\frac{r_ir_j}{r^2}f(r) + \left(\delta_{ij} - \frac{r_ir_j}{r^2}\right)g(r)\right], \tag{20}$$

where f and g are related by

$$g = f + \frac{1}{2} r \frac{\mathrm{d}f}{\mathrm{d}r}. \tag{21}$$

Here u_{rms} is the root-mean-square of any velocity component. The relation between $R_1, \ldots R_4$ and f and g in the isotropic case can be found by projection of (20) onto the four orthogonal tensors corresponding to each of $R_1, \ldots R_4$,

$$\frac{1}{u_{rms}^2} R_1 = \frac{z^2}{r^2} f + \frac{\rho^2}{r^2} g, \quad \frac{1}{u_{rms}^2} R_2 = \frac{\rho^2}{r^2} f + \frac{z^2}{r^2} g, \tag{22a,b}$$

$$\frac{1}{u_{rms}^2} R_3 = g, \quad \frac{1}{u_{rms}^2} R_4 = \frac{\rho z}{r^2} (f - g). \tag{22c,d}$$

It can readily be verified that both equations (19*a*) and (19*b*) in this case reduce to (21).

3.2. *Single-point limit of* $\boldsymbol{R}$

The single-point limit can be considered as a special case of the limit when $\boldsymbol{r}$ becomes parallel to $\boldsymbol{\lambda}$. In this limit $e^{(1)}$ and $e^{(2)}$ are not defined and $\boldsymbol{R}$ must become a tensor that can be written only in terms of $\boldsymbol{\lambda}$ and the Kronecker tensor. This can only be obtained if $R_2 = R_3$ and $R_4 = S_1 = S_2 = 0$, when ρ is equal to zero. The second and third terms of the correlation tensor will turn into the desired form by virtue of (15*a*).

Considering the symmetries of the correlation functions we obtain for small separations,

$$R_2 - R_3 \sim \rho^2, \quad R_4 \sim \rho z, \quad S_1 \sim \rho z, \quad S_2 \sim \rho^2. \tag{23}$$

3.3. *Fourier analysis of* $\boldsymbol{R}$

First we define the Fourier-transform of $\boldsymbol{R}$ in the standard way,

$$\widehat{R}_{ij}(\boldsymbol{k}) = \frac{1}{(2\pi)^3} \int R_{ij}(\boldsymbol{r}) \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) \mathrm{d}^3 r. \tag{24}$$

The spectrum tensor $\widehat{\boldsymbol{R}}$ is, just as $\boldsymbol{R}$, determined by three separate components, with the difference that the continuity condition is much simpler to apply in Fourier space. To determine $\widehat{\boldsymbol{R}}$ we choose the three components

$$\lambda_i \lambda_j \widehat{R}_{ij} = \frac{2}{(2\pi)^2} \int_0^\infty \int_0^\infty \rho R_1 \cos(k_z z) J_0(k_\rho \rho) \, \mathrm{d}z \, \mathrm{d}\rho, \tag{25a}$$

$$\begin{aligned} \tfrac{1}{2}(\delta_{ij} - \lambda_i \lambda_j) \widehat{R}_{ij} &= \frac{1}{(2\pi)^2} \int_0^\infty \int_0^\infty \rho (R_2 + R_3) \cos(k_z z) J_0(k_\rho \rho) \, \mathrm{d}z \, \mathrm{d}\rho \\ &= \frac{1}{(2\pi)^2} \int_0^\infty \int_0^\infty \rho [R_2 k_\rho \rho \cos(k_z z) J_1(k_\rho \rho) + R_4 k_z \rho \sin(k_z z) J_0(k_\rho \rho)] \mathrm{d}z \, \mathrm{d}\rho, \end{aligned} \tag{25b}$$

$$\lambda_i \epsilon_{jmn} \lambda_m k_n \widehat{R}_{ij} = -\frac{2}{(2\pi)^2} \int_0^\infty \int_0^\infty \rho S_1 k_\rho \sin(k_z z) J_1(k_\rho \rho) \mathrm{d}z \, \mathrm{d}\rho. \tag{25c}$$

Here J_n is the nth-order Bessel function (see Appendix B). In (25*b*) we have used equation (19*b*).

In a wind tunnel we can identify the z-direction with the direction of the mean flow. If we make use of Taylor's hypothesis which states that for low turbulence intensities the z-transformation can be replaced by a transformation in time of a measured signal, then equations (25*a,b*) imply that the full three-dimensional energy spectrum for axisymmetric turbulence without rotation can be determined by a simultaneous measurement of two velocity components u and v at two points at variable distance to each other on a line perpendicular to the mean flow. In the case with rotation equation (25c) implies that we must also measure u and w simultaneously. These measurements can be carried out by two hot-wire cross-probes, one of them fixed and the other one traversing a line perpendicular to the mean flow.

4. Helicity

Helicity is defined as the correlation between velocity and vorticity. The two-point helicity tensor can be defined as

$$H_{ij}(\boldsymbol{r}) = \langle u_i(O)\omega_j(P)\rangle = \epsilon_{jmn}\frac{\partial}{\partial r_m}R_{in}(\boldsymbol{r}). \tag{26}$$

As seen from (14) we not only have $R_{ij}(-\boldsymbol{r}) = R_{ji}(\boldsymbol{r})$ but also $R_{ij}(-\boldsymbol{r}) = R_{ij}(\boldsymbol{r})$, owing to reflectional symmetry in the planes normal to $\boldsymbol{\lambda}$. It follows that $H_{ij}(\boldsymbol{r}) = -H_{ij}(-\boldsymbol{r})$ and consequently

$$H_{ij}\,|_{r=0} = \boldsymbol{0}. \tag{27}$$

Hence the single-point correlation between velocity and vorticity is zero for homogeneous axisymmetric turbulence.

5. Dynamical equation for $\boldsymbol{R}$

The dynamical equation for $\boldsymbol{R}$ can be written (Hinze 1975)

$$\frac{\partial R_{ij}(\boldsymbol{r})}{\partial t} = -\frac{\partial U_k}{\partial x_s}A_{ksij}(\boldsymbol{r}) + T_{ij}(\boldsymbol{r}) + \Pi_{ij}(\boldsymbol{r}) - \varepsilon_{ij}(\boldsymbol{r}) \tag{28}$$

where $\partial U_k/\partial x_s$ is the mean flow gradient tensor, which for homogeneous turbulence must be constant, and

$$A_{ksij}(\boldsymbol{r}) = \delta_{ki}R_{sj}(\boldsymbol{r}) + \delta_{kj}R_{si}(-\boldsymbol{r}) + r_s\frac{\partial}{\partial r_k}R_{ij}(\boldsymbol{r}), \tag{29a}$$

$$T_{ij}(\boldsymbol{r}) = \frac{\partial}{\partial r_k}(M_{ikj}(\boldsymbol{r}) - M_{jki}(-\boldsymbol{r})), \tag{29b}$$

$$\Pi_{ij}(\boldsymbol{r}) = \frac{\partial}{\partial r_i}P_j(\boldsymbol{r}) - \frac{\partial}{\partial r_j}P_i(-\boldsymbol{r}), \tag{29c}$$

$$\varepsilon_{ij}(\boldsymbol{r}) = -2\nu\nabla^2 R_{ij}(\boldsymbol{r}). \tag{29d}$$

Here $\boldsymbol{T}$ is the transfer tensor, $\boldsymbol{M}$ is the two-point triple correlation tensor

$$M_{ikj}(\boldsymbol{r}) = \langle u_i(O)u_k(O)u_j(P)\rangle, \tag{30}$$

$\boldsymbol{\Pi}$ is the two-point pressure–strain tensor, $\boldsymbol{P}$ is the pressure–velocity correlation tensor

$$P_i(\boldsymbol{r}) = \frac{1}{\varrho}\langle p(O)u_i(P)\rangle, \tag{31}$$

and $\boldsymbol{\varepsilon}$ is the two-point dissipation tensor. Taking the single-point limit of equation (28) we obtain the Reynolds stress transport equation.

In the following we will study each of the terms in (28) and formulate them in terms of scalar functions, paying special attention to the problem of which quantities can be measured.

6. Mean flow gradient term

In the case of axisymmetry the mean flow gradient has the form

$$\frac{\partial U_i}{\partial x_s} = \tfrac{1}{2}\sigma(3\lambda_i\lambda_s - \delta_{is}) - \Omega\epsilon_{isn}\lambda_n, \tag{32}$$

where σ is the rate of strain along the direction of symmetry and Ω is the rate of rotation about the symmetry axis. The mean flow gradient term in the dynamical equation can now be expressed as

$$\begin{aligned}
-\frac{\partial U_k}{\partial x_s}A_{ksij}(\boldsymbol{r}) = {} & \lambda_i\lambda_j[(-2\sigma + \sigma\mathscr{D})R_1] + e_i^{(2)}e_j^{(2)}[(\sigma + \sigma\mathscr{D})R_2 + 4\Omega S_2] \\
& + e_i^{(1)}e_j^{(1)}[(\sigma + \sigma\mathscr{D})R_3 - 4\Omega S_2] \\
& + (\lambda_i e_j^{(2)} + \lambda_j e_i^{(2)})[(-\tfrac{1}{2}\sigma + \sigma\mathscr{D})R_4 + 2\Omega S_1] \\
& + (\lambda_i e_j^{(1)} + \lambda_j e_i^{(1)})[(-\tfrac{1}{2}\sigma + \sigma\mathscr{D})S_1 - 2\Omega R_4] \\
& + (e_i^{(2)}e_j^{(1)} + e_j^{(2)}e_i^{(1)})[(\sigma + \sigma\mathscr{D})S_2 + 2\Omega(R_3 - R_2)],
\end{aligned} \tag{33}$$

where we have introduced the differential operator

$$\mathscr{D} = \frac{1}{2}\rho\frac{\partial}{\partial\rho} - z\frac{\partial}{\partial z}. \tag{34}$$

The effect of the strain on the correlation functions is twofold. On the one hand it acts as a source or sink, depending on the sign of σ and which of the correlations is being considered. On the other hand it acts as a deformer of the correlation functions by virtue of the differential terms which are zero in the single-point limit and therefore conservative. The rotational terms are all zero in the single-point limit and therefore conservative as well. They introduce a coupling between the proper tensor terms and the skew tensor terms, showing that the skew tensors really are needed to describe axisymmetric turbulence subjected to mean rotation. The correlations S_1 and S_2 are not generally zero if there is a rotation about the symmetry axis.

7. Dissipation tensor

A straightforward calculation shows that the two-point dissipation tensor can be expressed as

$$\begin{aligned}
\varepsilon_{ij} = -2\nu \Bigg[& \lambda_i\lambda_j\nabla^2 R_1 + e_i^{(2)}e_j^{(2)}\left(\nabla^2 R_2 + \frac{2}{\rho^2}(R_3 - R_2)\right) \\
& + e_i^{(1)}e_j^{(1)}\left(\nabla^2 R_3 + \frac{2}{\rho^2}(R_2 - R_3)\right) + (\lambda_i e_j^{(2)} + \lambda_j e_i^{(2)})\left(\nabla^2 R_4 - \frac{1}{\rho^2}R_4\right) \\
& + (\lambda_i e_j^{(1)} + \lambda_j e_i^{(1)})\left(\nabla^2 S_1 - \frac{1}{\rho^2}S_1\right) + (e_i^{(1)}e_j^{(2)} + e_j^{(1)}e_i^{(2)})\left(\nabla^2 S_2 - \frac{4}{\rho^2}S_2\right)\Bigg].
\end{aligned} \tag{35}$$

The Fourier-transform of ε is by standard Fourier analysis

$$\widehat{\varepsilon}_{ij} = 2\nu k^2\widehat{R}_{ij}, \tag{36}$$

showing that the full three-dimensional dissipation spectrum can in principle be measured in the same way as the previously outlined measurement of the three-dimensional energy spectrum.

7.1. *Single-point limit of* ε

From (23) and (35) it can readily be shown that in the single-point limit there are two non-vanishing components of $\boldsymbol{\varepsilon}$, corresponding to the two non-vanishing components of $\boldsymbol{R}$. In order to express these components in terms of the curvatures at $|\, \boldsymbol{r} \,| = 0$ of the non-vanishing correlations we expand

$$R_1 = R_{1_0} - \alpha_1\rho^2 - \beta_1 z^2 + \ldots, \qquad (37a)$$
$$R_2 = R_{2_0} - \alpha_2\rho^2 - \beta_2 z^2 + \ldots, \qquad (37b)$$
$$R_3 = R_{2_0} - \alpha_3\rho^2 - \beta_3 z^2 + \ldots, \qquad (37c)$$
$$R_4 = \gamma\rho z + \ldots, \qquad (37d)$$

From (19*a,b*) it now follows that

$$\gamma = \beta_1, \quad \beta_3 = \beta_2, \quad \alpha_3 = 3\alpha_2 - \beta_1 \qquad (38)$$

and

$$\lambda_i\lambda_j\varepsilon_{ij}\,|_{r=0} = -2\nu\nabla^2 R_1\,|_{r=0} = 4\nu(2\alpha_1 + \beta_1), \qquad (39)$$

$$\begin{aligned} \tfrac{1}{2}(\delta_{ij} - \lambda_i\lambda_j)\varepsilon_{ij}\,|_{r=0} &= -2\nu\nabla^2(R_2 + R_3)_{r=0} \\ &= 4\nu(4\alpha_2 + \beta_2 - \beta_1) = \tfrac{4}{3}\nu(4\alpha_3 + 3\beta_3 + \beta_1). \end{aligned} \qquad (40)$$

Equivalent results, but in different form, were derived by Batchelor (1946) and Chandrasekhar (1950). The two single-point components of ε can be determined by measuring two velocity components, either u and v or u and w, on a line perpendicular to the mean flow. By making use of Taylor's hypothesis in the z-direction it is possible to construct the two-dimensional surfaces R_1 and R_2 or R_1 and R_3 from the measured data, and from them estimate the curvatures at $|\, \boldsymbol{r} \,| = 0$. With a very accurate measurement it is of course sufficient to determine these functions on the z and ρ axes only, but since the measurement involves some difficulties and since with the same effort we can obtain the whole of the correlation surfaces it seems nevertheless to be a better procedure to estimate the curvatures from the entire surfaces.

In the case of isotropy the functions f and g can be expanded as

$$f = 1 - \frac{1}{2\lambda^2}r^2 + \ldots, \quad g = 1 - \frac{1}{\lambda^2}r^2 + \ldots, \qquad (41)$$

where λ here is the Taylor microscale. From equations (22*a–d*) it is seen that when axisymmetry turns into isotropy

$$\beta_1 = \alpha_2 = u^2\frac{1}{2\lambda^2}, \quad \beta_2 = \alpha_1 = \beta_3 = \alpha_3 = u^2\frac{1}{\lambda^2}. \qquad (42)$$

This indicates that for cases which are not too extreme R_3 is a steeper function of ρ in the neighbourhood of zero than R_2 is, which implies that the best conditioned choice is to measure R_1 and R_2.

8. Triple correlation tensor

The third-order two-point triple correlation tensor $\boldsymbol{M}$ is symmetric in the first two indices and solenoidal,

$$\frac{\partial M_{ikj}}{\partial r_j} = 0, \qquad (43)$$

in the last index. In the axisymmetric case this tensor can be represented in a similar way to the second-order tensor,

$$
\begin{aligned}
M_{ikj}(\boldsymbol{r}) = {} & \lambda_i\lambda_k\lambda_j M_1 + e_i^{(2)}e_k^{(2)}\lambda_j M_2 + e_i^{(1)}e_k^{(1)}\lambda_j M_3 \\
& +\lambda_i\lambda_k e_j^{(2)} M_4 + e_i^{(2)}e_k^{(2)}e_j^{(2)} M_5 + e_i^{(1)}e_k^{(1)}e_j^{(2)} M_6 \\
& +(\lambda_i e_k^{(2)} + \lambda_k e_i^{(2)})\lambda_j M_7 + (\lambda_i e_k^{(2)} + \lambda_k e_i^{(2)})e_j^{(2)} M_8 \\
& +(\lambda_i e_k^{(1)} + \lambda_k e_i^{(1)})e_j^{(1)} M_9 + (e_i^{(2)}e_k^{(1)} + e_k^{(2)}e_i^{(1)})e_j^{(1)} M_{10} \\
& +\lambda_i\lambda_k e_j^{(1)} N_1 + e_i^{(2)}e_k^{(2)}e_j^{(1)} N_2 + e_i^{(1)}e_k^{(1)}e_j^{(1)} N_3 \\
& +(\lambda_i e_k^{(1)} + \lambda_k e_i^{(1)})\lambda_j N_4 + (\lambda_i e_k^{(1)} + \lambda_k e_i^{(1)})e_j^{(2)} N_5 \\
& +(e_i^{(2)}e_k^{(1)} + e_k^{(2)}e_i^{(1)})\lambda_j N_6 + (\lambda_i e_k^{(2)} + \lambda_k e_i^{(2)})e_j^{(1)} N_7 \\
& +(e_i^{(2)}e_k^{(1)} + e_k^{(2)}e_i^{(1)})e_j^{(2)} N_8.
\end{aligned} \tag{44}
$$

By demanding reflectional symmetry in the normal plane of $\boldsymbol{\lambda}$ we can conclude that M_4, M_5, M_6, M_7 and M_{10} are even in z, while the rest of the M-functions are odd in z. Reflectional symmetry also implies that N_1, N_2, N_3, N_4 and N_8 are even in z while N_5, N_6 and N_7 are odd in z. The N-terms are skew-tensors and must therefore be zero in the case of strong axisymmetry. From symmetry arguments we can, as in the case of the second-order correlation, conclude that some correlations are even and some are odd in ρ. In fact it turns out that triple correlations that are even in z are odd in ρ and *vice versa*. This can also be seen from the following argument.† If we demand that $\boldsymbol{M}$ shall be continuously differentiable at the origin, each of the terms in (44) must be even in ρ. The vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are odd in ρ, so the scalar functions multiplying an odd number of these two unit vectors must be odd and the functions multiplying an even number must be even in ρ. By inspection we find that the functions multiplying an even number of the two unit vectors are odd in z and the functions multiplying an odd number are even in z. We can now conclude that all the triple correlations and all their even-order derivatives are zero in the single-point limit.

Using (17) and (18) it is straightforward to derive the following equations from the continuity condition (43):

$$\frac{\partial}{\partial\rho}(\rho M_4) = -\rho\frac{\partial M_1}{\partial z}, \tag{45a}$$

$$M_{10} = \frac{1}{2}\left(\frac{\partial}{\partial\rho}(\rho M_5) + \rho\frac{\partial M_2}{\partial z}\right), \tag{45b}$$

$$M_{10} = -\frac{1}{2}\left(\frac{\partial}{\partial\rho}(\rho M_6) + \rho\frac{\partial M_3}{\partial z}\right), \tag{45c}$$

$$M_9 = \frac{\partial}{\partial\rho}(\rho M_8) + \rho\frac{\partial M_7}{\partial z}, \tag{45d}$$

$$N_7 = -\frac{\partial}{\partial\rho}(\rho N_5) - \rho\frac{\partial N_4}{\partial z}, \tag{45e}$$

$$N_3 - N_2 = \frac{\partial}{\partial\rho}(\rho N_8) + \rho\frac{\partial N_6}{\partial z}. \tag{45f}$$

We notice that the N_1 term alone fulfils the condition (43).

† The author wishes to thank Magnus Hallbäck for giving him this argument.

8.1. *Single-point behaviour of* $\boldsymbol{M}$

All the even-order derivatives of the triple correlations are zero in the single-point limit. It is possible to show that the single-point first-order derivatives of the first ten non-rotational correlation functions are zero as well.

Since M_1, M_2, M_3, M_8 and M_9 are odd in z their first-order single-point derivatives with respect to ρ are zero and since M_4, M_5, M_6, M_7 and M_{10} are odd in ρ their first-order single-point derivatives with respect to z are zero. Let x' and z' be Cartesian coordinates with z' in the direction of $\boldsymbol{\lambda}$. From the condition of homogeneity we have

$$\left.\frac{\partial M_1}{\partial z}\right|_{r=0} = \left\langle uu\frac{\partial u}{\partial z}\right\rangle = \frac{1}{3}\frac{\partial}{\partial z'}\langle uuu\rangle = 0, \tag{46}$$

$$\left.\frac{\partial M_5}{\partial \rho}\right|_{r=0} = \left\langle vv\frac{\partial v}{\partial \rho}\right\rangle = \frac{1}{3}\frac{\partial}{\partial x'}\langle vvv\rangle = 0. \tag{47}$$

The distinction between differentiation with respect to ρ and z and differentiation with respect to x' and z' is important here. The first type of differentiation involves a variation of the relative positions of two points, while the second type of differentiation involves a variation of the position of a single point. In a similar way we can show that

$$\frac{\partial}{\partial z}[2M_8 + M_2]_{r=0} = \frac{\partial}{\partial z}[2M_9 + M_3]_{r=0} = 0, \tag{48}$$

$$\frac{\partial}{\partial \rho}[2M_{10} + M_6]_{r=0} = \frac{\partial}{\partial \rho}[2M_7 + M_4]_{r=0} = 0. \tag{49}$$

These relations are compatible with the continuity equations (45*a–d*) only if the first-order derivatives of each of the correlation functions are zero, which easily can be verified. In a similar way we can show that the single-point first-order derivatives of each of the N-correlations are zero except for N_2 and N_8 for which we can only show that

$$\frac{\partial}{\partial \rho}[2N_8 + N_2]_{r=0} = 0. \tag{50}$$

To determine various terms in the single-point equation of ε we have to measure single-point third-order derivatives of the triple correlations. In order to clearly see how these derivatives are related to each other we make the expansions

$$M_1 = a_1 z^3 + b_1 z\rho^2 + \ldots, \quad M_2 = a_2 z^3 + b_2 z\rho^2 + \ldots, \tag{51a,b}$$

$$M_3 = a_3 z^3 + b_3 z\rho^2 + \ldots, \quad M_4 = c_4 \rho^3 + d_4 \rho z^2 + \ldots, \tag{51c,d}$$

$$M_5 = c_5 \rho^3 + d_5 \rho z^2 + \ldots, \quad M_6 = c_6 \rho^3 + d_6 \rho z^2 + \ldots, \tag{51e,f}$$

$$M_7 = c_7 \rho^3 + d_7 \rho z^2 + \ldots, \quad M_8 = a_8 z^3 + b_8 z\rho^2 + \ldots, \tag{51g,h}$$

$$M_9 = a_9 z^3 + b_9 z\rho^2 + \ldots, \quad M_{10} =_{10} \rho^3 + d_{10} \rho z^2 + \ldots. \tag{51i,j}$$

From equations (41*a–d*) we have the following relations:

$$b_1 = -4c_4, \quad a_1 = -\tfrac{2}{3}d_4, \tag{52a}$$

$$c_{10} = 2c_5 + \tfrac{1}{2}b_2 = -2c_6 - \tfrac{1}{2}b_3\,, \tag{52b}$$

$$d_{10} = d_5 + \tfrac{3}{2}a_2 = -d_6 - \tfrac{3}{2}a_3 \tag{52c}$$

$$a_9 = a_8, \quad b_9 = 3b_8 + 2d_7. \tag{52d}$$

Axisymmetry requires that M_2 is equal to M_3 when ρ is equal to zero, so that

$$a_2 = a_3. \tag{53}$$

There are twenty expansion coefficients in (51*a–j*). In (52*a–d*) there are eight independent equations relating these coefficients to each other, and (49) gives one extra relation. Hence there are eleven independent single-point third-order derivatives of the triple correlations in strongly axisymmetric turbulence, whereas there is only one for isotropic turbulence. In the isotropic case the triple correlation tensor can be written (Kármán & Howarth 1938, see also Hinze 1975)

$$M_{ikj} = u_{rms}^3 \left[\frac{k - rk'}{2r^3} r_i r_k r_j + \frac{2k + rk'}{4r} (r_i \delta_{kj} + r_k \delta_{ij}) - \frac{k}{2r} r_j \delta_{ik} \right]. \tag{54}$$

Here $k(r)$ is the triple correlation between three velocity components all oriented in the direction of $\boldsymbol{r}$. For small separations k can be expanded,

$$k = \tau r^3 + \ldots . \tag{55}$$

The relation between the coefficients in (51*a–j*) and this single coefficient τ when axisymmetry turns into isotropy is readily found to be

$$a_1 = c_5 = \tau, \quad b_1 = d_5 = 2\tau, \tag{56a,b}$$
$$a_2 = a_3 = b_3 = c_4 = c_6 = d_6 = -\tfrac{1}{2}\tau, \tag{56c}$$
$$b_2 = d_4 = -\tfrac{3}{2}\tau, \quad b_8 = d_7 = \tfrac{1}{4}\tau, \tag{56d,e}$$
$$a_8 = a_9 = b_9 = c_7 = c_{10} = d_{10} = \tfrac{5}{4}\tau. \tag{56f}$$

8.2. *Transfer tensor*

For homogeneous turbulence the transfer tensor $\boldsymbol{T}$ is zero in the single-point limit (Batchelor 1946). Its role is essentially to transfer energy from large to small scales. It is a second-order tensor that fulfils the index symmetry condition (2). In the axisymmetric case it can thus be written in terms of six scalars two of which are zero in the case of strong axisymmetry,

$$T_{ij}(\boldsymbol{r}) = \lambda_i \lambda_j T_1 + e_i^{(2)} e_j^{(2)} T_2 + e_i^{(1)} e_j^{(1)} T_3 + (\lambda_i e_j^{(2)} + \lambda_j e_i^{(2)}) T_4 \\ + (\lambda_i e_j^{(1)} + \lambda_j e_i^{(1)}) U_1 + (e_i^{(2)} e_j^{(1)} + e_j^{(2)} e_i^{(1)}) U_2. \tag{57}$$

A straightforward calculation shows that

$$T_1 = 2\frac{\partial M_1}{\partial z} + \frac{2}{\rho}\frac{\partial}{\partial \rho}(\rho M_7), \tag{58a}$$
$$T_2 = \frac{2}{\rho}\frac{\partial}{\partial \rho}(\rho M_5) - \frac{2}{\rho} M_6 + 2\frac{\partial M_8}{\partial z} - \frac{2}{\rho} M_{10}, \tag{58b}$$
$$T_3 = \frac{2}{\rho} M_6 + 2\frac{\partial M_9}{\partial z} + \frac{2}{\rho} M_{10} + \frac{2}{\rho}\frac{\partial}{\partial \rho}(\rho M_{10}), \tag{58c}$$
$$T_4 = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho M_2) - \frac{1}{\rho} M_3 + \frac{\partial M_4}{\partial z}, \tag{58d}$$
$$U_1 = \frac{\partial N_1}{\partial z} + \frac{1}{\rho} N_5 + \frac{\partial N_4}{\partial z} + \frac{2}{\rho} N_6 + \frac{\partial N_6}{\partial \rho} + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho N_7), \tag{58e}$$
$$U_2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho N_2) - \frac{2}{\rho} N_3 + \frac{\partial N_5}{\partial z} + \frac{\partial N_7}{\partial z} + \frac{3}{\rho} N_8 + \frac{\partial N_8}{\partial \rho}. \tag{58f}$$

Equation (45d) has been used to reduce formula (58d). The trace of the Fourier-transform $\widehat{\boldsymbol{T}}$ of $\boldsymbol{T}$ is the well-known 'transfer-function' which in the case of isotropic turbulence gives us a picture of the nonlinear 'cascade' of energy from large to small scales. In axisymmetric turbulence we have in the single-point limit two directions, the longitudinal which is parallel to $\boldsymbol{\lambda}$ and the transverse which is perpendicular to $\boldsymbol{\lambda}$. The two corresponding diagonal components of $\widehat{\boldsymbol{T}}$ describe the nonlinear transfer of energy within each component. Since $\boldsymbol{T}$ is equal to zero in the single-point limit, each component of $\widehat{\boldsymbol{T}}$ will integrate to zero and thus 'conserve energy'. It is interesting to see that both these components can be expressed in terms of correlations that are indeed possible to measure:

$$\lambda_i\lambda_j\widehat{T}_{ij} = \frac{2}{(2\pi)^2}\int_0^\infty\int_{-\infty}^\infty \rho[M_1k_z\sin(k_zz)J_0(k_\rho\rho) + M_7k_\rho\cos(k_zz)J_1(k_\rho\rho)]\,\mathrm{d}z\,\mathrm{d}\rho, \quad (59a)$$

$$\begin{aligned}\tfrac{1}{2}(\delta_{ij}-\lambda_i\lambda_j)\widehat{T}_{ij} = \frac{1}{(2\pi)^2}\int_0^\infty\int_{-\infty}^\infty &\rho[M_2\tfrac{1}{2}k_\rho k_z\rho\sin(k_zz)J_1(k_\rho\rho)\\ &+ M_5\cos(k_zz)[\tfrac{1}{4}k_\rho^2\rho(J_2(k_\rho\rho)-J_0(k_\rho\rho)) + \tfrac{1}{2}k_\rho J_1(k_\rho\rho)]\\ &- M_7k_z^2\rho\cos(k_zz)J_0(k_\rho\rho) + M_8k_\rho k_z\rho\sin(k_zz)J_1(k_\rho\rho)]\,\mathrm{d}z\,\mathrm{d}\rho. \quad (59b)\end{aligned}$$

For the derivation of (59b) we have used (45b) and (45d).

If we identify the M-functions in (59a,b) as the measurable correlations

$$\left.\begin{aligned} &M_1 = \langle u(O)u(O)u(P)\rangle, \quad M_2 = \langle v(O)v(O)u(P)\rangle, \quad M_5 = \langle v(O)v(O)v(P)\rangle,\\ &M_7 = \langle u(O)v(O)u(P)\rangle, \quad M_8 = \langle u(O)v(O)v(P)\rangle, \end{aligned}\right\} \quad (60)$$

we see that if we again use Taylor's hypothesis for the transformation in the z-direction, it is sufficient to simultaneously measure u and v at two points at variable distance on a line perpendicular to the mean flow to determine the transfer spectra. The transfer spectra and the energy spectra can thus be determined from the same measurement.

9. Pressure terms

The pressure–velocity correlation tensor $\boldsymbol{P}$ is a first-order solenoidal tensor (or vector),

$$\frac{\partial P_i}{\partial r_i} = 0. \quad (61)$$

In the axisymmetric case it can be written in terms of three scalar functions,

$$P_i(\boldsymbol{r}) = \lambda_iP_1 + e_i^{(2)}P_2 + e_i^{(1)}Q_1, \quad (62)$$

where Q_1 is zero in strong axisymmetry. The symmetry constraints require that P_2 and Q_1 are even in z and odd in ρ and that P_1 is odd in z and even in ρ. Hence $\boldsymbol{P}$ is equal to zero in the single-point limit. The last term in (62) automatically fulfils the continuity condition while P_1 and P_2 are coupled by the equation

$$\frac{\partial}{\partial\rho}(\rho P_2) = -\rho\frac{\partial P_1}{\partial z}. \quad (63)$$

The two-point pressure–strain tensor occurring in the dynamical equation can be expressed as

$$\begin{aligned}\Pi_{ij}(\boldsymbol{r}) = {} & \lambda_i\lambda_j 2\frac{\partial P_1}{\partial z} + e_i^{(2)}e_j^{(2)}2\frac{\partial P_2}{\partial \rho} \\ & + e_i^{(1)}e_j^{(1)}\frac{2}{\rho}P_2 + (\lambda_i e_j^{(2)} + \lambda_j e_i^{(2)})\left(\frac{\partial P_1}{\partial \rho} + \frac{\partial P_2}{\partial z}\right) \\ & + (\lambda_i e_j^{(1)} + \lambda_j e_i^{(1)})\frac{\partial Q_1}{\partial z} + (e_i^{(1)}e_j^{(2)} + e_j^{(1)}e_i^{(2)})\left(\frac{\partial}{\partial \rho} - \frac{1}{\rho}\right)Q_1. \end{aligned} \tag{64}$$

In the single-point limit there is only one independent component of $\boldsymbol{\Pi}$. This component is responsible for a transfer of energy between the longitudinal and transverse velocity components.

9.1. *Single-point limit of* $\boldsymbol{\Pi}$

From the dynamical equation and the continuity condition the Poisson equation for the pressure–velocity correlation tensor can be derived,

$$\nabla^2 P_j(\boldsymbol{r}) = 2\frac{\partial U_i}{\partial x_s}\frac{\partial}{\partial r_i}R_{sj}(\boldsymbol{r}) - \frac{\partial^2}{\partial r_i \partial r_k}M_{ikj}(\boldsymbol{r}). \tag{65}$$

In the case of axisymmetry two independent scalar equations can be derived from (65), one for the non-rotational part of the pressure–velocity correlation (P_1 or P_2) and one for the extra rotational part (Q_1). Here we are mainly interested in the first part since it is the one which contributes to the non-vanishing component of $\boldsymbol{\Pi}$ in the single-point limit. Projecting (65) onto $\boldsymbol{\lambda}$, using (32), (19*a*) and (44) and taking the partial derivative with respect to z we obtain

$$\nabla^2\frac{\partial P_1}{\partial z} = \frac{\partial F}{\partial z}, \tag{66}$$

where

$$F = 3\sigma\frac{\partial R_1}{\partial z} - 2\Omega\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho S_1) - \frac{\partial^2 M_1}{\partial z^2} - \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial M_2}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial M_3}{\partial \rho} - \frac{2}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial M_7}{\partial z}\right). \tag{67}$$

The meaning of σ and Ω is given by (32) in §6. The boundary condition is of course that P_1 and all its derivatives vanish as $|\boldsymbol{r}| \to \infty$. The solution of this equation can be found from the method of sources,

$$\frac{\partial P_1}{\partial z}(\boldsymbol{r}) = -\frac{1}{4\pi}\int \frac{\partial F}{\partial z}(\boldsymbol{r}')\frac{\mathrm{d}^3 r'}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{68}$$

Taking the single-point limit of (68) we obtain

$$\begin{aligned}\lambda_i\lambda_j\,\Pi_{ij}\big|_{\boldsymbol{r}=0} = 2\frac{\partial P_1}{\partial z}\bigg|_{|\boldsymbol{r}|=0} = {} & \int_0^\infty\int_{-\infty}^\infty \rho r^{-1}\left[-3\sigma\frac{\partial^2 R_1}{\partial z^2} + 2\Omega\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial S_1}{\partial z}\right)\right. \\ & \left. + \frac{\partial^3 M_1}{\partial z^3} + \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial^2 M_2}{\partial \rho \partial z}\right) - \frac{1}{\rho}\frac{\partial^2 M_3}{\partial \rho \partial z} + \frac{2}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial^2 M_7}{\partial z^2}\right)\right]\mathrm{d}z\,\mathrm{d}\rho, \end{aligned} \tag{69}$$

where $r = (\rho^2 + z^2)^{1/2}$. Dividing this expression into a 'rapid' part and a 'slow' part we obtain after partial integration,

$$\lambda_i\lambda_j\,\Pi_{ij}^{(rapid)}\Big|_{\boldsymbol{r}=0} = \int_0^\infty\int_{-\infty}^\infty 3\rho r^{-5}[-\sigma(2z^2 - \rho^2)R_1 + 2\Omega\rho z S_1]\,\mathrm{d}z\,\mathrm{d}\rho, \tag{70a}$$

$$\lambda_i\lambda_j\,\Pi_{ij}^{(slow)}\Big|_{r=0} = \int_0^\infty\int_{-\infty}^{\infty} 3\rho r^{-5}[z(2z^2-3\rho^2)r^{-2}M_1 + z(M_2-M_3) + z(3\rho^2-2z^2)r^{-2}M_2 + 2\rho(4z^2-\rho^2)r^{-2}M_7]\,\mathrm{d}z\,\mathrm{d}\rho. \tag{70b}$$

Here we have used that, on the basis of symmetry, M_2 is equal to M_3 when ρ is equal to zero. By virtue of the continuity relations, each of the integrals can be expressed in alternative ways (see Appendix C). The terms in the slow pressure-strain integral are grouped in such a way that each of them can be shown to integrate to zero in the isotropic case.

Close to the origin the triple correlations are of $O(r^3)$, which shows the convergence of the slow pressure–strain integral. The lowest-order term in (70*a*) will integrate to zero which easily can be shown by a change to spherical coordinates. This shows the convergence of the rapid pressure-strain integral. The small-scale contribution to the integrals will grow quadratically with the distance from the origin, if we integrate from zero and outward in both the variables. Small errors in the correlations at small scales should therefore not have any disastrous effects on the integrals. On large scales there is a damping factor of $O(r^{-2})$ for the rapid part, and for the slow part a damping factor of $O(r^{-3})$, multiplying the correlations in the integrands. This may be sufficiently well-behaved for an experimental determination of the pressure–strain integrals.

To determine the purely strain-related part of the rapid pressure–strain it is sufficient to measure u simultaneously at two points at variable distance from each other on a line perpendicular to the mean flow. This can be done with two single hot-wire probes. To measure the rotational part of the rapid pressure-strain we have to measure u at one point and w at another point. This can be carried out with one single and one cross-probe.

To see what is required to measure the slow pressure–strain we identify the functions which must be determined as

$$\left.\begin{aligned} M_1 &= \langle u(O)u(O)u(P)\rangle, \quad M_2 = \langle v(O)v(O)u(P)\rangle, \\ M_3 &= \langle w(O)w(O)u(P)\rangle, \quad M_7 = \langle u(O)v(O)u(P)\rangle. \end{aligned}\right\} \tag{71}$$

Each of these correlations can be measured by two hot-wire probes, one fixed cross-probe and one single probe traversing a line perpendicular to the mean flow. It is also possible to measure all of them at the same time, using two cross-probes. In the mean flow direction we again use Taylor's hypothesis.

9.2. *Pressure–strain spectrum*

The Poisson equation for the pressure–strain can also be solved in Fourier-space, giving us the pressure–strain spectrum. Taking the Fourier transform of (66) we obtain after partial integration

$$\lambda_i\lambda_j\widehat{\Pi}_{ij}^{(rapid)} = -\frac{2k_z}{(2\pi)^2k^2}\int_0^\infty\int_{-\infty}^{\infty}\rho[-3\sigma R_1 k_z\cos(k_z z)J_0(k_\rho\rho) - 2\Omega S_1 k_\rho \sin(k_z z)J_1(k_\rho\rho)]\,\mathrm{d}z\,\mathrm{d}\rho, \tag{72a}$$

$$\lambda_i\lambda_j\widehat{\Pi}_{ij}^{(slow)} = -\frac{2k_z}{(2\pi)^2k^2}\int_0^\infty\int_{-\infty}^{\infty}\rho[M_1k_z^2\sin(k_z z)J_0(k_\rho\rho) + M_2k_\rho^2\tfrac{1}{2}\sin(k_z z)[J_0(k_\rho\rho)-J_2(k_\rho\rho)] + M_3k_\rho\rho^{-1}\sin(k_z z)J_1(k_\rho\rho) + M_7k_\rho k_z 2\cos(k_z z)J_1(k_\rho\rho)]\,\mathrm{d}z\,\mathrm{d}\rho. \tag{72b}$$

Here we have used the fact that $M_2 = M_3$ when ρ is equal to zero.

10. Single-point equations for $\boldsymbol{R}$ and $\boldsymbol{\varepsilon}$

It is now straightforward to derive equations for each of the scalar functions defining $\boldsymbol{R}$. Projecting the dynamical equation (32) onto the six orthogonal tensors formed from the three unit vectors we obtain six equations of which three are independent. Here we restrict ourselves with stating the single-point equations for $\boldsymbol{R}$ and $\boldsymbol{\varepsilon}$. The two non-vanishing components of $\boldsymbol{R}$ are $R_{1_0} = \langle uu \rangle$ and $R_{2_0} = R_{3_0} = \langle vv \rangle = \langle ww \rangle$. The two corresponding components of $\boldsymbol{\varepsilon}$, (39) and (40), we denote by ε_{1_0} and ε_{2_0}. The single-point equations are

$$\frac{\partial R_{1_0}}{\partial t} = -2\sigma R_{1_0} - \varepsilon_{1_0} + 2\left.\frac{\partial P_1}{\partial z}\right|_{\boldsymbol{r}=\boldsymbol{0}}, \tag{73}$$

$$\frac{\partial R_{2_0}}{\partial t} = \sigma R_{2_0} - \varepsilon_{2_0} - \left.\frac{\partial P_1}{\partial z}\right|_{\boldsymbol{r}=\boldsymbol{0}}, \tag{74}$$

$$\begin{aligned}\frac{\partial \varepsilon_{1_0}}{\partial t} &= -2\sigma\varepsilon_{1_0} - 2\nu\sigma[\nabla^2\mathscr{D}R_1]_{\boldsymbol{r}=\boldsymbol{0}} \\ &\quad -4\nu^2[\nabla^4 R_1]_{\boldsymbol{r}=\boldsymbol{0}} - 2\nu\left[\nabla^2\left(T_1 + 2\frac{\partial P_1}{\partial z}\right)\right]_{\boldsymbol{r}=\boldsymbol{0}},\end{aligned} \tag{75}$$

$$\begin{aligned}\frac{\partial \varepsilon_{2_0}}{\partial t} &= \sigma\varepsilon_{2_0} - 2\nu\sigma[\nabla^2\mathscr{D}\tfrac{1}{2}(R_2+R_3)]_{\boldsymbol{r}=\boldsymbol{0}} \\ &\quad -2\nu^2[\nabla^4(R_2+R_3)]_{\boldsymbol{r}=\boldsymbol{0}} - 2\nu\left[\nabla^2\left(\tfrac{1}{2}(T_2+T_3) - \frac{\partial P_1}{\partial z}\right)\right]_{\boldsymbol{r}=\boldsymbol{0}},\end{aligned} \tag{76}$$

where $\mathscr{D}$ is the differential operator defined in (34). It should be possible to measure each of the terms in (73) and (74). The dissipative term and the pressure–strain term in each of the two equations are surely somewhat difficult to measure. Which of these two is the most difficult to measure we can only find out by experiment. In equations (75) and (76) it is clear that the destruction terms must be very difficult to measure. To determine these terms experimentally would require a measurement of the single-point fourth-order derivatives of the correlation functions. There are six independent fourth-order derivatives of the purely axisymmetric correlation functions, of which two are mixed derivatives. This can be seen from the symmetries of the correlation functions and from equations (19*a,b*). A direct determination of these derivatives must be considered as almost impossible with experimental techniques used today. The mean strain-related terms can be measured in the same way as the dissipative terms in (73) and (74), and it is possible that the time derivatives in (75) and (76) can be measured with some accuracy as well. Whether or not it is possible to measure the single-point Laplacian of the transfer term and the pressure–strain term is hard to say. This would require a measurement of single-point third-order derivatives of the triple correlations. Since all lower-order derivatives are zero we cannot exclude the possibility of measuring these terms with some accuracy.

If we make the expansion

$$S_1 = s\rho z + \dots \tag{77}$$

it is possible to express the right-hand sides of (75) and (76), except for the destruction terms, in terms of expansion coefficients which have been defined for the correlation functions. This has already been done for ε_{1_0} and ε_{2_0} in (39) and (40), and consequently for the first right-hand-side term of each equation. The other terms can (except for a multiplying factor) be expressed as

$$\nabla^2\mathscr{D}R_1\,|_{\boldsymbol{r}=\boldsymbol{0}} = -4(\alpha_1 - \beta_1), \tag{78}$$

$$\nabla^2 \mathscr{D} \tfrac{1}{2}(R_2 + R_3)|_{r=0} = -2(\alpha_2 + \alpha_3 - 2\beta_2), \tag{79}$$

$$\nabla^2 T_1 \;|_{r=0} = 12a_1 + 8b_1 + 32c_7 + 8d_7, \tag{80}$$

$$\begin{aligned} \nabla^2 \tfrac{1}{2}(T_2 + T_3)|_{r=0} &= 16c_5 + 4d_5 + 6a_8 + 4b_8 + 6a_9 + 4b_9 + 16c_{10} + 4d_{10} \\ &= 6a_2 + 8b_2 + 48c_5 + 8d_5 + 8d_7 + 12a_8 + 16b_8, \end{aligned} \tag{81}$$

$$\nabla^2 \left.\frac{\partial P_1}{\partial z}\right|_{r=0} = -6\sigma\beta_1 - 4\Omega s - 6a_1 - 6b_2 + 2b_3 - 8d_7. \tag{82}$$

11. Summary and conclusions

A representation of two-point correlation tensors of homogeneous axisymmetric turbulence has been found, such that each measurable correlation corresponds to a single scalar function, and moreover such that the equations of continuity take the most simple form. Apart from reproducing known results in a simpler and more comprehensible form, such as the expressibility of the dissipation tensor in terms of four expansion coefficients corresponding to the Taylor microscale of isotropic turbulence, the analysis produces a number of new results. The two-point triple correlation tensor is analysed, and it is proved that the leading-order terms of the triple correlations are of $O(r^3)$ for small separations. From symmetry arguments and the equations of continuity it is deduced that the number of independent single-point third-order derivatives of the axisymmetric triple correlations is eleven, and the relations between these and the relevant terms in the equation for the dissipation tensor are stated. A scalar Poisson equation for the pressure–strain is derived, and the single-point solution is written as a sum of integrals over measurable correlations. This is perhaps the strongest result, since it suggests that the rapid pressure–strain and the slow pressure–strain can be individually determined by a direct measurement. In general it is argued that the axisymmetric correlations can be determined by measuring components of the velocity at two points at variable distance from each other on a line perpendicular to the mean flow in a wind tunnel. Using the Fourier–Bessel transform it is shown that the three-dimensional energy, transfer and pressure–strain Fourier spectra can be obtained by such a measurement.

Despite the improvement of theory all the hard work is yet to be done. Although the new analysis illuminates some qualitative aspects of the dynamics of axisymmetric turbulence, such as the crucial role of the skew correlations when rotation is present, it deals almost exclusively with the kinematical problem. The more difficult dynamical problem is left for the future, and so is the task of experiment. It is our hope that this work will stimulate experimental progress in the field of turbulence. The analysis offers some tools to experimentalists interested in the fundamental questions of turbulence, such as Kolmogorov's hypothesis of isotropy of small scales. It is indeed a vast challenge to investigate the decay and the return to isotropy of turbulence downstream of an axisymmetric contraction in a wind tunnel, to determine the pressure–strain and the dissipation tensor, to study the transfer of energy between components and the distribution of anisotropy among different scales, and to further investigate the effects of rotation on all these quantities.

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Appendix A. The method of invariants and dyadic representation of the axisymmetric two-point correlations

A dyad (see for example Gibbs 1948) can be formed as the tensor-product or outer-product between two vectors. Dyads form a vector space of dimension $3^2 = 9$, and are thus second-order tensors. The only difference between dyads and traditional tensors is in notation. A dyad can be multiplied with a vector, either from the left or from the right, using either the dot- or the cross-product. The cross-product gives us a new dyad and the dot-product gives us a vector. A dyadic representation of the second-order velocity correlation tensor is very natural, since it is formed as the mean value of the tensor-product between two velocity vectors,

$$\boldsymbol{R} = \langle \boldsymbol{u}(O)\boldsymbol{u}(P)\rangle. \tag{A 1}$$

The axisymmetric two-point correlation dyad can be written

$$\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{r}, \boldsymbol{\lambda}), \tag{A 2}$$

where $\boldsymbol{r}$ and $\boldsymbol{\lambda}$ are defined as in §3. Again it is important to remember that $\boldsymbol{r}$ is defined as a separation vector between the two points O and P, and not as a position vector of a single point, and thus it is coordinate-system independent. The symmetry condition (2) of §1 can be written

$$\boldsymbol{R}^T(\boldsymbol{r}, \boldsymbol{\lambda}) = \boldsymbol{R}(-\boldsymbol{r}, \boldsymbol{\lambda}), \tag{A 3}$$

where the transpose $\boldsymbol{R}^T$ of $\boldsymbol{R}$ is the dyad satisfying

$$\boldsymbol{a}\cdot\boldsymbol{R}^T\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{R}\cdot\boldsymbol{a} \tag{A 4}$$

for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Invariant theory (Robertson 1940) tells us that the projection of $\boldsymbol{R}$ onto any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ must be a function of the invariants of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{r}$ and $\boldsymbol{\lambda}$, that is

$$\boldsymbol{a}\cdot\boldsymbol{R}\cdot\boldsymbol{b} = F(I_1, I_2, \ldots), \tag{A 5}$$

where the invariants $I_1, I_2, \ldots$ are the scalars that can be formed from the vectors, using the cross- and the dot-product, for example $\boldsymbol{r}\cdot\boldsymbol{r}$, $\boldsymbol{r}\cdot\boldsymbol{b}$, $(\boldsymbol{a}\times\boldsymbol{r})\cdot\boldsymbol{\lambda}$ and $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{\lambda}$. The most general form of $\boldsymbol{R}$ consistent with (A5) is a sum of all linearly independent dyads that can be formed from the unit dyad $\boldsymbol{U}$ and the vectors $\boldsymbol{r}$ and $\boldsymbol{\lambda}$, where each of these dyads multiplies a scalar function of $\boldsymbol{r}\cdot\boldsymbol{r}$ and $\boldsymbol{r}\cdot\boldsymbol{\lambda}$. The operations by which we can form the dyads are the outer-product and the cross-product. Thus we can for example form $\boldsymbol{U}$, $\boldsymbol{\lambda}\boldsymbol{r}$, $\boldsymbol{\lambda}\times\boldsymbol{U}$ and $(\boldsymbol{\lambda}\times\boldsymbol{r})\boldsymbol{r}$. Instead of representing $\boldsymbol{R}$ directly by these dyads, which corresponds to Batchelor's (1946) approach, we first form the two unit vectors $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ as normalized cross-products as in (8) of §3. $\boldsymbol{R}$ can now be represented by the dyads that can be formed from our three orthogonal unit vectors, since they span three-dimensional space. The fact that the unit dyad is no longer needed once we have our unit vector base, can be seen from the relation

$$\boldsymbol{U} = \boldsymbol{\lambda}\boldsymbol{\lambda} + \boldsymbol{e}^{(1)}\boldsymbol{e}^{(1)} + \boldsymbol{e}^{(2)}\boldsymbol{e}^{(2)}, \tag{A 6}$$

which holds for any orthogonal unit vector base. Applying condition (A3) and demanding reflectional symmetry in planes normal to $\boldsymbol{\lambda}$, we find the representation

$$\boldsymbol{R} = \boldsymbol{\lambda\lambda} R_1 + \boldsymbol{e}^{(2)}\boldsymbol{e}^{(2)} R_2 + \boldsymbol{e}^{(1)}\boldsymbol{e}^{(1)} R_3 + (\boldsymbol{\lambda e}^{(2)} + \boldsymbol{e}^{(2)}\boldsymbol{\lambda}) R_4 + (\boldsymbol{\lambda e}^{(1)} + \boldsymbol{e}^{(1)}\boldsymbol{\lambda}) S_1 + (\boldsymbol{e}^{(2)}\boldsymbol{e}^{(1)} + \boldsymbol{e}^{(1)}\boldsymbol{e}^{(2)}) S_2, \quad \text{(A 7)}$$

which exactly corresponds to the Cartesian notation (14) of §3. In dyadic notation we can write the relations (17) and (18) as

$$\nabla \boldsymbol{e}^{(1)} = -\frac{1}{\rho}\boldsymbol{e}^{(1)}\boldsymbol{e}^{(2)}, \quad \nabla \boldsymbol{e}^{(2)} = \frac{1}{\rho}\boldsymbol{e}^{(1)}\boldsymbol{e}^{(1)}, \quad \text{(A 8)}$$

$$\boldsymbol{\lambda}\cdot\nabla = \frac{\partial}{\partial z}, \quad \boldsymbol{e}^{(2)}\cdot\nabla = \frac{\partial}{\partial \rho}, \quad \boldsymbol{e}^{(1)}\cdot\nabla = 0, \quad \text{(A 9)}$$

and the continuity relation (3) of §1, as

$$\nabla\cdot\boldsymbol{R} = \nabla\cdot\boldsymbol{R}^T = \boldsymbol{0}. \quad \text{(A 10)}$$

Applying this condition to (A7) and using (A8) and (A9) we find

$$\boldsymbol{\lambda}\left(\frac{\partial R_1}{\partial z} + \frac{1}{\rho}R_4 + \frac{\partial R_4}{\partial \rho}\right) + \boldsymbol{e}^{(2)}\left(\frac{1}{\rho}R_2 + \frac{\partial R_2}{\partial \rho} - \frac{1}{\rho}R_3 + \frac{\partial R_4}{\partial z}\right) + \boldsymbol{e}^{(1)}\left(\frac{\partial S_1}{\partial z} + \frac{2}{\rho}S_2 + \frac{\partial S_2}{\partial \rho}\right) = \boldsymbol{0} \quad \text{(A 11)}$$

which constitutes three separate equations that can also be written in the form of (19*a–c*) of §3.

The advantage of the dyadic representation is that it is more geometrical in character than the Cartesian tensor representation, and that the coordinate-system independence is totally displayed. From a typographical point of view it is also cleaner since no indices are needed. But there is a major drawback of the dyadic representation that would have been even more clearly seen if we had represented the triple correlations as triads, which of course is possible. This drawback is the lack of commutativity, when different operations are to be performed on dyads and triads. Taking derivatives will be a heavy task. For this reason and for historical reasons we have adopted the Cartesian tensor representation. It is easy to see that there is a one-to-one correspondence between the two representations. The choice between them is therefore a question of notational convenience.

Appendix B. Fourier–Bessel transform

When expressing three-dimensional Fourier transforms of cylindrical symmetric correlations in terms of Bessel-functions we use the representation

$$J_n(x) = \frac{\mathrm{i}^{-n}}{2\pi}\int_0^{2\pi} \exp(\mathrm{i}x\cos\phi)\cos(n\phi)\,\mathrm{d}\phi. \quad \text{(B 1)}$$

For example

$$\frac{1}{(2\pi)^3}\int R_1(\rho,z)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}^3r$$
$$= \frac{1}{(2\pi)^3}\int_0^\infty\int_{-\infty}^\infty\int_0^{2\pi} \rho R_1(\rho,z)\exp(-\mathrm{i}k_z z)\exp(-\mathrm{i}k_\rho\rho\cos\phi)\,\mathrm{d}\phi\,\mathrm{d}\rho\,\mathrm{d}z$$
$$= \frac{1}{(2\pi)^3}\int_0^\infty\int_{-\infty}^\infty R_1(\rho,z)\cos(k_z z)J_0(k_\rho\rho)\,\mathrm{d}z\,\mathrm{d}\rho, \quad \text{(B 2)}$$

where we also have used the fact that R_1 is even in z. Transforms of derivatives of the correlations can be expressed in terms of higher-order Bessel functions if we use partial integration. For example

$$\frac{1}{(2\pi)^3}\int (R_2+R_3)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}^3r$$

$$=\frac{1}{(2\pi)^3}\int_0^\infty\int_{-\infty}^\infty\int_0^{2\pi}\rho\left[2R_2+\rho\left(\frac{\partial R_2}{\partial\rho}+\frac{\partial R_4}{\partial z}\right)\right]\exp(-\mathrm{i}k_z z-\mathrm{i}k_\rho\rho\cos\phi)\,\mathrm{d}\phi\,\mathrm{d}z\,\mathrm{d}\rho$$

$$=-\frac{1}{(2\pi)^3}\int_0^\infty\int_{-\infty}^\infty\int_0^{2\pi}\rho[-\rho\mathrm{i}k_\rho\cos\phi R_2-\rho\mathrm{i}k_z R_4]\exp(-\mathrm{i}k_z z-\mathrm{i}k_\rho\rho\cos\phi)\,\mathrm{d}\phi\,\mathrm{d}z\,\mathrm{d}\rho$$

$$=\frac{1}{(2\pi)^2}\int_0^\infty\int_{-\infty}^\infty\rho[R_2k_\rho\rho\cos(k_z z)J_1(k_\rho\rho)+R_4k_z\rho\sin(k_z z)J_0(k_\rho\rho)]\,\mathrm{d}z\,\mathrm{d}\rho, \qquad \text{(B 3)}$$

where we have used the properties that R_2 and $R_4\to 0$ as $|\boldsymbol{r}|\to\infty$ and that R_2 is even and R_4 is odd in z. In a similar way the integrals of the transfer spectrum and the pressure-strain spectrum can be simplified.

Appendix C. Alternative forms of pressure–strain integrals

By using the continuity equations (19*a–c*) and (45*a–d*), the different terms in the single-point solution (69) of the Poisson equation can be expressed in alternative forms to that already given in (70*a–b*),

$$-3\sigma\int_0^\infty\int_{-\infty}^\infty\rho r^{-1}\frac{\partial^2R_1}{\partial z^2}\,\mathrm{d}z\,\mathrm{d}\rho=3\sigma\int_0^\infty\int_{-\infty}^\infty\rho^2zr^{-5}R_4\,\mathrm{d}z\,\mathrm{d}\rho, \qquad \text{(C 1a)}$$

$$2\Omega\int_0^\infty\int_{-\infty}^\infty r^{-1}\frac{\partial}{\partial\rho}(\rho\frac{\partial S_1}{\partial z})\,\mathrm{d}z\,\mathrm{d}\rho=-2\Omega\int_0^\infty\int_{-\infty}^\infty 3\rho^3r^{-5}S_2\,\mathrm{d}z\,\mathrm{d}\rho, \qquad \text{(C 1b)}$$

$$\int_0^\infty\int_{-\infty}^\infty\rho r^{-1}\frac{\partial^3M_1}{\partial z^3}\,\mathrm{d}z\,\mathrm{d}\rho=3\int_0^\infty\int_{-\infty}^\infty\rho^2r^{-7}(\rho^2-4z^2)M_4\,\mathrm{d}z\,\mathrm{d}\rho, \qquad \text{(C 1c)}$$

$$\int_0^\infty\int_{-\infty}^\infty\rho r^{-1}\left[\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial^2M_2}{\partial\rho\partial z}\right)-\frac{1}{\rho}\frac{\partial^2M_3}{\partial\rho\partial z}\right]\mathrm{d}z\,\mathrm{d}\rho$$

$$=3\int_0^\infty\int_{-\infty}^\infty\rho r^{-5}[2zM_2+\rho(M_5+M_6)]+\rho zr^{-7}(3\rho^2-2z^2)M_2\,\mathrm{d}z\,\mathrm{d}\rho \qquad \text{(C 1d)}$$

$$=3\int_0^\infty\int_{-\infty}^\infty\rho^2r^{-7}(4z^2-\rho^2)(M_5+M_6)-\rho zr^{-7}(3\rho^2-2z^2)M_3$$

$$-\rho r^{-5}[2zM_3+\rho(M_5+M_6)]\mathrm{d}z\,\mathrm{d}\rho, \qquad \text{(C 1e)}$$

$$2\int_0^\infty\int_{-\infty}^\infty r^{-1}\frac{\partial}{\partial\rho}(\rho\frac{\partial^2M_7}{\partial z^2})\,\mathrm{d}z\,\mathrm{d}\rho$$

$$=6\int_0^\infty\int_{-\infty}^\infty\rho r^{-5}(M_9-M_8)+z\rho r^{-7}(2z^2-3\rho^2)M_8\,\mathrm{d}z\,\mathrm{d}\rho. \qquad \text{(C 1f)}$$

The terms in the integrals of the slow pressure–strain are grouped in a way such that each of them can be shown to integrate to zero in the isotropic case.

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